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Method of inverse dynamic systems and its application for recovering internal heat sources

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Abstract—The method of the inversion of linear distributed dynamic systems is developed which allows one to conduct a qualitative study and to obtain an analytical solution of a certain class of inverse heat conduction problems. As an illustration of the general approach, the inverse problem of recovering the time-varying component of an internal heat source is considered.

1. INTRODUCTION

THE PROBLEMS of the inversion of linear distributed dynamic systems (DSs) are connected with the recovery of inputs of a DS from the measured functionals being determined in instantaneous states of the system. Interpreted in thermophysical terms, these are the problems concerned with the recovery of time-varying internal and boundary conditions of heat transfer: external heat fluxes, volumetric sources, etc. The common approach to the solution of such inverse problems (IPs) consists in their reduction to the first-kind Volterra equations with subsequent application of regularization techniques [1, 2]. The other formalism is based on the inverse system being preliminarily represented in the space of states. This representation also presupposes the use of regularization techniques, but alongside makes it possible to answer a number of questions of qualitative character: to divide the inverse system into correct and incorrect parts; to determine whether the correct part of the inverse DS is stable over an infinite time interval; to indicate the least volume of prior information about initial conditions which would be sufficient to recover the inputs.

Two techniques are now available for constructing distributed inverse systems: the method of structural factorization [3–5] and the method of recalculation of boundary conditions [6–8]. In this paper, the method of structural factorization is developed in application to the solution and qualitative investigation of an IP concerned with determining the function of internal heat sources.

2. STATEMENT OF THE PROBLEM

First, the statement of the problem of DS inversion will be given in the form which allows one to cover a wide class of IP.

Using the methods of functional analysis, a linearly distributed system can be described [9] by a system of equations of the form

$$\Omega:\begin{cases} w_t = Aw+Bu(t), \quad w(0) = w_0, \quad w_t = \dfrac{\partial w}{\partial t} \\ y(t) = Rw \end{cases}$$

where w is the state of the DS Ω; $u(t)$ and $y(t)$ are the input and output; $A\colon H \to H$ the infinitesimal operator of the semigroup e^{At} of class C^0 [9] acting in a Hilbert space of states of the DS Ω; $B\colon U \to H$ the linear operator determined over the space U of input values; $R\colon H \to Y$ the linear operator which characterizes the manner in which the state of the DS Ω is observed; Y the space of output values.

For the given initial state w_0 the system Ω induces the input/output map

$$y(t) = R\,e^{At}\,w_0+R\int_0^t e^{A(t-s)}\,Bu(s)\,ds. \qquad (1)$$

The problem, which leads to the concept of an inverse DS, consists in the recovery of the input $u(t)$ over the given time interval $[0, t_0]$ ($t_0 \leqslant \infty$) from the measured output $y(t)$ over the same time interval. The DS Ω^{-1} is the inverse to the DS Ω when it recovers the input $u(\tau)$, $\tau \in [0, t]$ by the output $y(t)$, $\tau \in [0, t]$ of the system Ω.

The abstract statement of the problem of DS inversion will be specified on the example of an IP concerned with the determination of the time-varying component of an internal heat source from measured temperatures at a certain inner point of a thermal system. The corresponding direct DS in a one-dimensional, linear approximation has the form

$$\Omega_1:\begin{cases} c(x)T_t = (\lambda(x)T_x)_x+b(x)u(t), \quad T(x,0) = T_0(x) & (2) \\ l_1 T \equiv T(0,t)\cos\alpha-\lambda(0)T_x(0,t)\sin\alpha = 0 & (3) \\ l_2 T \equiv T(l,t)\cos\beta-\lambda(l)T_x(l,t)\sin\beta = 0 & (4) \\ y(t) = T(x_0,t) \equiv \displaystyle\int_0^l \delta(x-x_0)T(x,t)\,dx. & (5) \end{cases}$$

NOMENCLATURE

$b(x)u(t)$	power density of volumetric heat sources
$c(x)$	specific heat
F	infinitesimal operator of dynamic systems (DSs) Ω^{-1}, Ω_r^{-1}
G	Green function of DS Ω_r^{-1}
H	Hilbert space of states of DS Ω
$L_2(0,l;c(x)\,dx)$	Hilbert space of functions the square of which can be integrated over segment $[0,l]$ with measure $c(x)\,dx$ (space of states of DSs Ω_1, Ω_r)
$r(x)$	weight function
$\|r\|$	norm of function r, $\sqrt{\langle r,r\rangle}$
T	temperature field (state of systems Ω_1, Ω_r)
t	time
$u(t)$	input of DSs $\Omega, \Omega_1, \Omega_r$
$\langle v,z\rangle$	scalar product of functions v,z in space $L_2(0,l;c(x)\,dx)$, $\int_0^l v(s)\bar{z}(s)c(s)\,ds$
w	state of DS Ω
x	space coordinate
$y(t)$	output of DSs $\Omega, \Omega_1, \Omega_r$.

Greek symbols

$\lambda(x)$	thermal conductivity
$\mu_1, \mu_2, \ldots$	eigenvalues of DS Ω_r^{-1} (spectrum of operator F)
$\bar{\mu}$	conjugate-complex value of μ
Ω	abstract linear dynamic system
Ω_1, Ω_r	systems describing direct heat conduction problems
Ω^{-1}	system inverse to DS Ω
Ω_r^{-1}	system inverse to DS Ω_r.

For the DS Ω_1 the generating operator A acts in the Hilbert space $H = L_2(0,l;c(x)\,dx)$ of functions the square of which can be integrated over $[0,l]$ with the measure $c(x)\,dx$ according to

$$Af = c^{-1}(\lambda f')' \quad (f' = df/dx). \tag{6}$$

The domain $D(A)$ of A consists of absolutely continuous functions that satisfy the conditions $l_1 f = l_2 f = 0$, $Af \in H$. The operator B coincides with the operator of multiplication by the function $c^{-1}b$, while R coincides with the functional which simulates temperature measurement at the point x_0; $Y = U = \mathbb{R}$ where $\mathbb{R}$ is the Euclidean space of real numbers.

Thus, the IPs of heat conduction associated with the determination of the time-varying component of internal heat sources, are incorporated into a more general class of problems concerned with the inversion of distributed DSs.

3. CONSTRUCTION OF AN INVERSE SYSTEM

To construct a system which would be inverse to the DS Ω, the space of states A will be expanded into a direct sum

$$H = H_1 \oplus H_2 \tag{7}$$

so that the arbitrary element w, $w \in H$, would admit the single representation $w = v + z$, $v \in H_2$, $z \in H_1$, where H_2 is the space formed by a set of solutions of the homogeneous equation $Rw = 0$ (in other words, H_2 represents the kernel of operator R). Equation (7) induces the projectors $P: H \to H_1$, $Q: H \to H_2$ defined by the equalities $Pw = z$, $Qw = v$. Using the familiar properties ($PQ = QP = 0$, $P^2 = P$, $Q^2 = 0$) of projection operators, the DS can be written in an equivalent form as

$$\Omega: \begin{cases} v_t = QAv + QAz + QBu, \quad v(0) = Qw_0 & (8) \\ z_t = PAv + PAz + PBu, \quad z(0) = Pw_0 & (9) \\ y = Rz. & (10) \end{cases}$$

Since $z \in H_1$, the homogeneous equation $Rz = 0$ has only the trivial solution $z = 0$. Therefore, z is determined uniquely from equation (10): $z = R^{-1}y$. Substituting $z = R^{-1}y$ into equations (8) and (9), a reduced DS is obtained

$$\tilde{\Omega}: \begin{cases} v_t = QAv + QAR^{-1}y + QBu, \quad v(0) = Qw_0 \\ y_1(t) \equiv R^{-1}y_t - PAR^{-1}y = PAv + PBu. \end{cases} \tag{11}$$

Under the assumption that the operator $PB: U \to H_1$ is invertible, the system of equations (11) yields the representation of an inverse DS in the space of states H_1

$$\Omega^{-1} = \tilde{\Omega}^{-1}: \begin{cases} v_t = Q(A - B(PB)^{-1}PA)v \\ \qquad + QAR^{-1}y(t) + QB(PB)^{-1}y_1(t) \\ u(t) = -(PB)^{-1}PAv + (PB)^{-1}y_1(t) \\ v(0) = Qw_0. \end{cases} \tag{12}$$

The invertibility of the operator PB is equivalent to that of RB and is a sufficient condition for the solution of IPs to be unique. Note, that in applied problems the operator RB is generally invertible.

The formal construction of the system Ω^{-1} given above can be rigorously substantiated under certain additional restrictions imposed on the DS Ω. Specifically, it is sufficient to suppose [3] that the operators R, B, $(PB)^{-1}$ are continuous and that the operator $PA: H \to H_1$ admits closure up to a continuous operator.

4. QUALITATIVE INVESTIGATION OF AN INVERSE SYSTEM

The study of the inverse DS Ω^{-1} allows certain qualitative conclusions to be made about IPs. Thus, for example, the position of the spectrum of the operator $F = Q(A-B(PB)^{-1}PA)$ on a complex plane characterizes the dynamic properties of an inverse system. These properties (asymptotic stability in the sense of Lyapunov [10], monotonicity of transient processes, etc.) are important for numerical and analogue solutions of IPs.

An analytical representation of the solution for IP is defined in terms of the subgroup e^{Ft} and has the form

$$u(t) = -(PB)^{-1}PA\Big(e^{Ft}v(0) + \int_0^t e^{F(t-s)}(QAR^{-1}y(s) + QB(PB)^{-1}y_1(s))\,ds\Big) + (PB)^{-1}y_1(t) \quad (13)$$

where

$$y_1(t) = R^{-1}y_t - PAR^{-1}y.$$

The structure of solution (13) is such that it allows the recovery of the input $u(t)$ at the rate of information supply about the output $y(t)$, i.e. in a real time scale. The property of incorrectness, usually inherent in IPs, shows up in solution (13) and is due to the required single differentiation of the observed value of $y(t)$. The second potential source of the irregularity of specific IPs is associated with the possible instability of the inverse DS Ω^{-1} in the sense of Lyapunov. As is known [11], in this case the errors of the initial data accumulate exponentially in numerical calculations of the IP solution in a real time scale.

Another aspect of the qualitative investigation of IPs is associated with the reduction of initial data for the problem. Since, according to equations (12), the initial state $v(0)$ of the inverse system Ω^{-1} is the projection of the initial state of the direct system onto the kernel of the operator R, then the condition $v(0) = Qw_0$ should be interpreted as a reduction of information about w_0. A further reduction can be made by studying the observed properties of a DS.† In fact, if the DS is not quite observable, then, according to ref. [9] the homogeneous equation

$$R e^{At} w_0 = 0, \quad t \geqslant 0 \quad (14)$$

has a non-trivial solution $w_0 \neq 0$. Let S be the operator which projects the space H onto the subspace M orthogonal with respect to the set of solutions of equation (14). M is called the reduced space of the DS Ω states. As is shown in ref. [9], the DS Ω can be written in the form equivalent to that of equation (1)

$$\Omega: \begin{cases} w_t = SAw + SBu, & w \in M \\ y = Rw, & w(0) = Sw_0. \end{cases} \quad (15)$$

Consequently, the condition $w(0) = Sw_0$ eliminates from equation (1) the excessive information about the initial state w_0.

Repeating the construction of the inverse system in application to equation (1), it can be obtained that

$$\Omega^{-1}: \begin{cases} v_t = QS(A-B(PSB)^{-1}PSA)v \\ \qquad + QSAR^{-1}y(t) + QSB(PSB)^{-1}y_1(t) \\ u(t) = -(PSB)^{-1}PSAv + (PSB)^{-1}y_1(t) \\ v(0) = PSw_0 \end{cases}$$

where $y_1(t) = R^{-1}y_t - PSAR^{-1}y$, P the operator which projects the space M onto the space of solutions of the equation $Rw = 0$, $w \in M$.

Note should be taken of the specific case of DS Ω^{-1} when the equality $PSA = 0$ is fulfilled. Then, the expression $u(t) = (PSB)^{-1}y_1(t)$ can be considered as an explicit solution of IP. It is independent of the state w and, consequently, of the state w of the direct DS Ω. A detailed analysis of this situation and corresponding problems of heat conduction IPs are given in refs. [5, 13]. Another example of the heat conduction IP, the solution of which is independent of the initial state of the direct DS, is the well-known problem [14] of determining the densities of heat fluxes on the body boundary from the measured temperatures and heat fluxes on its opposite boundary.

5. RECOVERY OF INTERNAL SOURCES

Consider the IP formulated earlier and dealing with the determination of the non-stationary component $u(t)$ of an internal source. Since the operator

$$RT = \int_0^l \delta(x-x_0)T(x,t)\,dx$$

which corresponds to the system Ω_1, is unbounded in the space $L_2(0,l;c(x)\,dx)$, the algorithm of structural factorization cannot be applied directly to the DS Ω_1. Therefore, the generalized function $\delta(x-x_0)$ in expression (5) is approximated by rather a smooth function with the carrier confined in the vicinity of point x_0. This means that the output of the DS Ω_1 should be described by

$$y(t) = \langle r, T\rangle = \int_0^l c(x)r(x)T(x,t)\,dx \quad (16)$$

where $r(x)$ is rather a smooth function satisfying the boundary conditions $l_1 r = l_2 r = 0$. It is natural to call the quantity $r(x)$ the weight function.

† A formal definition of the observability comes to the following: the DS Ω is called to be observable when the conditions $u(t) \equiv 0$, $y(t) \equiv 0$ at $t \in [0, \infty]$ yield the equality $w(0) = 0$. At present, the concept of observability, which first came into being in the theory of automatic control [12], is fundamentally important for studying the systems of different origins.

The set of equations (2)–(4) and (16) determine a DS which will be designated by the symbol Ω_r. For the DS Ω_r the space H_1 coincides with the one-dimensional space stretched over the weight function r, and the operators P, Q, PA, PB act according to

$$Pf = \|r\|^{-2}r\langle r,f\rangle, \quad Qf = f-Pf,$$

$$PAf = \|r\|^{-2}r\int_0^l r(\lambda f_x)_x\,dx$$

$$\equiv \|r\|^{-2}r\int_0^l(\lambda r_x)_x f\,dx = \|r\|^{-2}r\langle Ar,f\rangle$$

$$PBu = \|r\|^{-2}r\langle c^{-1}b,r\rangle u = k\|r\|^{-2}ru$$

$$k = \langle c^{-1}b,r\rangle = \int_0^l b(x)r(x)\,dx \tag{17}$$

where $\|r\| = \sqrt{\langle r,r\rangle}$ is the norm of the function r in the space $L_2(0,l;c(x)\,dx)$; operator A is defined by formula (6).

Assuming $z = \gamma(t)r$, $\gamma\in\mathbb{R}$ and taking into account relations (17), the analogue of equations (8)–(10) for the DS (2)–(4) and (16) is obtained

$$v_t = Av-\|r\|^{-2}r\langle Ar,v\rangle+(Ar-\|r\|^{-2}r\langle Ar,r\rangle)\gamma+(c^{-1}b-\|r\|^{-2}rk)u(t)$$

$$r\dot{\gamma} = r(\|r\|^{-2}\langle Ar,v\rangle+\|r\|^{-2}\langle Ar,r\rangle\gamma+\|r\|^{-2}ku)$$

$$y = \|r\|^2\gamma \quad (\dot{\gamma} = d\gamma/dt). \tag{18}$$

When $k\neq 0$, then $u(t)$ can be determined from equations (18). This will yield an explicit representation of the inverse DS

$$cv_t = (\lambda v_x)_x-k^{-1}b\int_0^l(\lambda r_x)_x v\,dx+b_1(x)y(t)+b_2(x)\dot{y}(t) \tag{19}$$

$$l_1 v = l_2 v = 0,$$

$$v(x,0) = v_0 = w_0-\|r\|^{-2}r\int_0^l crw_0\,dx \tag{20}$$

$$u(t) = -k^{-1}\left(\int_0^l(\lambda r_x)_x v\,dx+\|r\|^{-2}\int_0^l(\lambda r_x)_x r\,dx\,y(t)-\dot{y}(t)\right) \tag{21}$$

where

$$b_1(x) = \|r\|^{-2}(\lambda r_x)_x-k^{-1}b\int_0^l(\lambda r_x)_x\,dx$$

$$b_2(x) = k^{-1}b-\|r\|^{-2}cr.$$

Note that incorporation of $v(\cdot,t)\in H_2$, $\forall\, t>0$ will result in

$$\langle v(\cdot,t),r\rangle = 0, \quad \forall\, t>0;$$

$$\langle c^{-1}b_1,r\rangle = 0, \quad \langle c^{-1}b_2,r\rangle = 0.$$

In specific inverse DSs the last two equalities can be conveniently used to control the calculations of coefficients $b_1(x)$, $b_2(x)$ and the identity $\langle v(\cdot,t),r\rangle\equiv 0$ can be used for controlling the numerical realization of the IP.

Let the Green function of the boundary-value problem, equations (19) and (20), be denoted by $G(x,\xi,t)$. Then, it will follow from equations (19)–(21) that the solution of the IP can be written in the form similar to that of equation (13)

$$u(t) = -k^{-1}\left(\int_0^l\int_0^l(\lambda r_x)_x G(x,\xi,t)\times c(\xi)v_0(\xi)\,d\xi\,dx+\int_0^t\int_0^l\int_0^l(\lambda r_x)_x\times G(x,\xi,t-\tau)f(\xi,\tau)\,d\xi\,dx\,d\tau+\|r\|^{-2}\int_0^l(\lambda r_x)_x r\,dx\,y-\dot{y}\right)$$

where

$$f(\xi,\tau) = b_1(\xi)y(\tau)+b_2(\xi)\dot{y}(\tau).$$

The Green function G will be given in terms of the spectral data of mutually conjugated boundary-value problems

$$\begin{cases}\mu c\tilde{v} = (\lambda\tilde{v}_x)_x-k^{-1}b\langle Ar,\tilde{v}\rangle, & \tilde{v} = \tilde{v}(x,\mu)\\ l_1\tilde{v} = l_2\tilde{v} = 0\end{cases} \tag{22}$$

$$\begin{cases}\bar{\mu}c\tilde{w} = (\lambda\tilde{w}_x)_x-k^{-1}Ar\langle b,\tilde{w}\rangle\\ l_1\tilde{w} = l_2\tilde{w} = 0\end{cases} \tag{23}$$

where μ is the spectral parameter, and $\bar{\mu}$ the conjugate-complex value of μ.

Introducing the following notation:

$$\mu_1,\mu_2,\ldots,\mu_i,\ldots \tag{24}$$

the spectrum of the infinitesimal operator F: $Fz = Az-k^{-1}c^{-1}\langle Ar,z\rangle$, $z\in H_2$ (the set of the eigenvalues of problem (22) in the order of decreasing real $\mathrm{Re}\,\mu_i$); $\{v_1(x),\ldots,v_i(x),\ldots\}$ and $\{w_1(x),\ldots,w_i(x),\ldots\}$ are corresponding sets of eigenfunctions of boundary-value problems (22) and (23); $\varphi(x,\mu)$ is the solution of the Cauchy problem

$$(\lambda\varphi_x)_x-\mu c\varphi = 0, \quad \varphi(0,\mu) = \sin\alpha,$$

$$\lambda(0)\varphi_x(0,\mu) = \cos\alpha;$$

$\psi(x,\mu)$ is the solution of the Cauchy problem

$$(\lambda\psi_x)_x-\mu c\psi = 0, \quad \psi(l,\mu) = \sin\beta,$$

$$\lambda(l)\psi_x(l,\mu) = \cos\beta.$$

By analogy with the classical Sturm–Liouville theory [15], it is possible to prove the following proposition which describes the spectral data of boundary-value problems (22) and (23).

Proposition

(1) The spectrum of operator F coincides with the set of roots of the characteristic equation

$$\theta(\mu) = \int_0^l \int_0^l r(x)c(x)g(x,\xi,\mu)b(\xi)\,d\xi\,dx$$
$$\equiv \mu^{-1}\left(k\Delta(\mu) - \int_0^l \int_0^l (\lambda r_x)_x \times g(x,\xi,\mu)b(\xi)\,d\xi\,dx\right) = 0$$

where

$$\Delta(\mu) = \varphi(l,\mu)\cos\beta - \lambda(l)\varphi_x(l,\mu)\sin\beta$$
$$\equiv \psi(0,\mu)\cos\alpha - \lambda(0)\psi_x(0,\mu)\sin\alpha$$
$$g(x,\xi,\mu) = \begin{cases}\varphi(x,\mu)\psi(\xi,\mu), & x \leqslant \xi\\ \psi(x,\mu)\varphi(\xi,\mu), & x > \xi.\end{cases}$$

(2) When $\theta(\mu_i) = 0$ the eigenfunctions, corresponding to μ_i, of boundary-value problems (22) and (23) have the form

$$v_i(x) = \int_0^l g(x,\xi,\mu_i)b(\xi)\,d\xi, \quad i = \overline{1,\infty} \quad (25)$$

$$w_i(x) = \int_0^l g(x,\xi,\bar{\mu}_i)(\lambda r_\xi)_\xi\,d\xi, \quad i = \overline{1,\infty}. \quad (26)$$

To represent the Green function G in explicit form, note that, according to ref. [16], the set of eigen- and associated functions of boundary-value problem (22) form the Riesz basis in the space $L_2(0,l;c(x)\,dx)$. Therefore, in the case of zero multiplicity of eigenvalues (24), function G can be determined from

$$G(x,\xi,t) = \sum_{i=1}^{\infty} \frac{v_i(x)\bar{w}_i(\xi)}{\langle w_i, v_i\rangle} e^{\mu_i t}. \quad (27)$$

Generally, when there are eigenvalues of non-zero multiplicity, function G can also be written in the form similar to that of equation (27), using for the purpose the eigen- and associated functions [16].

6. EXAMPLE

As an example, find the Green function for the inverse DS, equations (19)–(21), with constant thermophysical parameters c and λ, boundary conditions $l_1T \equiv T(0,t) = 0$, $l_2T \equiv T(l,t) = 0$, source function $b(x)u(t) = b_1u(t)$, $b_1 = \text{const.}$ and weight function $r(x) = x(l-x)$.

A corresponding direct DS is not observable and therefore, it admits the reduction of the state space. Proceeding just as in ref. [8], it can be easily verified that the reduced space of states consists of axially symmetric functions of the form

$$T^+(x,t) = \frac{T(x,t)+T(l-x,t)}{2}.$$

The characteristic equation

$$\theta(\mu) = \mu^{-1}\left(4\left(\frac{c\mu}{\lambda}\right)^{-3/2}\operatorname{sh}\frac{1}{2}\left(\frac{c\mu}{\lambda}\right)^{1/2} + \frac{l^3}{6} - 2l\left(\frac{c\mu}{\lambda}\right)^{-1}\operatorname{ch}\frac{l}{2}\left(\frac{c\mu}{\lambda}\right)^{1/2}\right) = 0$$

of an inverse system has only real negative roots

$$\mu_n = -\frac{4\lambda\psi_n}{l^2c}, \quad n = 1,2,\ldots \quad (28)$$

where ψ_n, $n = \overline{1,\infty}$, are positive roots of

$$\operatorname{tg}\psi - \psi - \tfrac{1}{3}\psi^3 = 0 \quad (29)$$

which are arranged in order of increasing magnitude.

The roots of equation (29) are approximated, with good accuracy, by the expression $\psi_n \simeq \pi(2n+1)/2$, therefore $\mu_n \simeq \lambda\pi^2(2n+1)/l^2c$. Since in the present specific case boundary-value problems (22) and (23) are self-conjugate, their eigenfunctions, corresponding to the values $\mu = \mu_1$, coincide. Calculation by either equation (25) or (26) shows that

$$v_n(x) = w_n(x) = \left(\sin\psi_n\frac{x-l}{l}\right)\sin\frac{\psi_n x}{l}. \quad (30)$$

The normalizing factor has the form

$$\langle v_n, w_n\rangle = cl\left(\frac{1}{4} + \frac{1}{\psi_n}(2\sin\psi_n - \sin 2\psi_n) + 2\cos 2\psi_n\right). \quad (31)$$

Thus, all the data required to represent the Green function in the form of equation (27) have been calculated in the explicit form of equations (28), (30) and (31).

Since in the present example the spectrum of the generating operator for the inverse DS consists of the real negative numbers (28), the inverse system is asymptotically stable in the Lyapunov sense, while the transient processes have a monotonically decaying character. Also note that eigenfunctions (30) form the Riesz basis in the space $L_2^+(0,l;c\,dx) \ominus r$ consisting of the functions which are symmetric with respect to $l/2$ and orthogonal with respect to the function $r(x) = x(l-x)$. This is in accord with the fact that the space $L_2^+(0,l;c\,dx) \ominus r$ can be taken as the minimal space of states of an inverse DS. The projection of the initial state $T_0(x)$ onto the space $L_2^+(0,l;c\,dx) \ominus r$ represents the minimal amount of information about the initial state of the inverse DS which is necessary for an IP to be solved.

Some results of a check on the numerical calculation of an IP at $b = 1$, $l = 1$, $c = 1$, $\lambda = 1$ are presented in Figs. 1 and 2. An inhomogeneous integral differential equation, which is associated with an inverse DS, was solved by the finite-difference method using an implicit scheme. The integral term of the equation

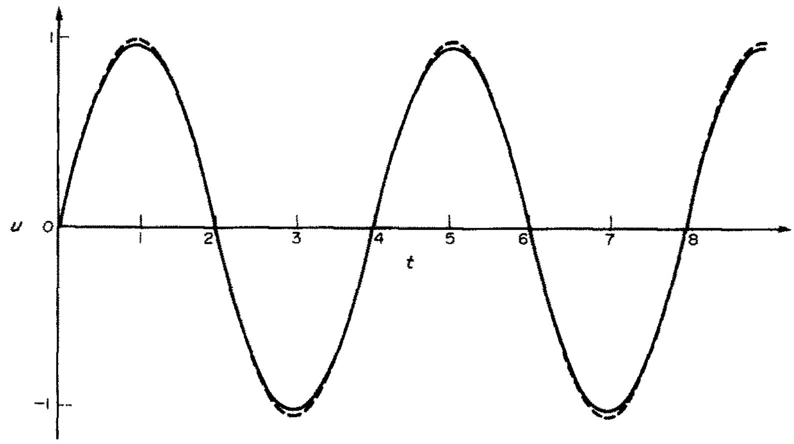

FIG. 1. Numerical simulation of inverse problems (IPs): - - - -, diagram of the function $u(t) = 5 \sin \pi t/4$; ———, result of the function $u(t) = 5 \sin \pi t/4$ recovery from the output $y(t)$.

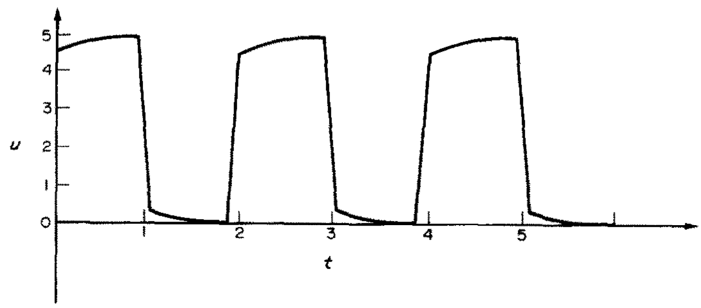

FIG. 2. Numerical simulation of IPs: ———, result of the recovery of the function

$$u(t) = \begin{cases} 5 \text{ at } t \in [0,2] \cup [4,6] \cup [8,10] \ldots \\ 0 \text{ at } t \in [2,4] \cup [6,8] \cup [10,12], \ldots \end{cases}$$

on each time layer was calculated from the solution obtained on the previous layer. Moreover, to increase the accuracy on each time layer an iteration process was organized which stopped at the level of the relative residual from the integral term of the equation. The differentiation of the output $y(t)$ of a direct DS was made with the aid of the EC computer service subprogram. No special simulation was made of the output $y(t)$ noising.

The results obtained make it possible to claim that the proposed method for solving IPs admits stable numerical realization.

7. CONCLUSIONS

Summarizing, it is possible to say that rather a general method of solution and qualitative investigation into the linear class of IPs with distributed parameters is suggested in the paper.

It is useful to verify on the first stage of an IP investigation whether the initial system Ω has the property of observability. If the DS Ω is not fully observable, the reduction of the space of its states is possible. This reduces the required volume of information about the initial conditions of the problem.

Depending on the character of an IP the construction of an inverse system may be based either on the algorithm of structural factorization, or the recalculation of boundary-value conditions. In the former case, it is sometimes necessary to somewhat alter the initial statement of an IP. For example, the δ-function, which simulates temperature measurement at a point, should be replaced by an approximating smooth function. Note, that this replacement can be conveniently made taking into account the averaging operation of temperature probes.

The method of inverse DSs makes it possible to employ, for studying the properties of IPs, the well-known concepts of the qualitative theory of differential equations, such as the quality of transition processes, stability and instability in the sense of Lyapunov. Moreover, the construction of an inverse DS usually entails an additional reduction of the required volume of information about the initial conditions of an IP.

After having represented the inverse system in the space of states, both numerical and analytical methods can be used for the actual solution of an IP. In the present work, the solution of the IP is given in terms of the Green function, and an example of numerical realization of the IP solution is given.

The property of the IP incorrectness is displayed in the analytical representation of the solution and is associated with the necessity for single differentiation of the signal measured. In the present work, the algorithm of the structural factorization is considered in the general situation, when the operator PB is reversible. In a degenerate case it is also possible to employ this algorithm [3], but then it is necessary to multiply differentiate the output of the original DS Ω.

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APPENDIX

Proof of the proposition

(1) The equation $(\mu-F)z = 0$, which determines the spectrum (24) and the eigenfunctions of operator F will be written in the form

$$(\mu-A)z = -k^{-1}c^{-1}b\langle z, Ar\rangle, \quad \langle z, r\rangle = 0. \qquad (A1)$$

First, consider the case when $\mu\notin\sigma(A)$. Then system (A1) is equivalent to the system

$$z = -k^{-1}(\mu-A)^{-1}c^{-1}\gamma \qquad (A2)$$

$$\gamma = \langle z, Ar\rangle \qquad (A3)$$

$$\langle z, r\rangle = 0. \qquad (A4)$$

If function r is the eigenfunction for operator A, then, according to equations (19)–(21), the inverse DS Ω_r^{-1} has the simplest form $u(t) = -k^{-1}(\|r\|^{-2}\langle r, Ar\rangle y - \dot{y})$. Therefore, it will be assumed that inequality $Ar \neq \mu r$, $\forall \mu\in\mathbb{C}$ is fulfilled and then $\gamma \neq 0$. By substituting z from equation (A2) into equations (A1) and (A4) and taking into account the inequality $\gamma \neq 0$, the system of equations results for the spectral parameter μ

$$k+\langle(\mu-A)^{-1}c^{-1}b, Ar\rangle = 0$$

$$\langle(\mu-A)^{-1}c^{-1}b, r\rangle = 0.$$

Using the equality $k = \langle b, c^{-1}r\rangle$ and the well-known property $(\mu-A)^{-1}z = \mu^{-1}(z+(\mu-A)^{-1}z)$, $\forall z\in D(A)$ [9], of the linear operator resolvent, it can be easily checked that

$$k+\langle(\mu-A)^{-1}c^{-1}b, Ar\rangle \equiv \mu\langle(\mu-A)^{-1}c^{-1}b, r\rangle.$$

Consequently, the system of equations (A1) at $\mu\notin\sigma(A)$ is equivalent to

$$0 = \mu^{-1}(k+\langle(\mu-A)^{-1}c^{-1}b, Ar\rangle) \equiv \langle(\mu-A)^{-1}c^{-1}b, r\rangle. \qquad (A5)$$

According to the theory of the Sturm–Liouville operators [15]

$$(\mu-A)^{-1}c^{-1}b = \frac{1}{\Delta(\mu)}\int_0^l g(x,\xi,\mu)b(\xi)\,\mathrm{d}\xi. \qquad (A6)$$

It follows directly from relations (A5) and (A6) that at $\mu\notin\sigma(A)$ the characteristic equation $\theta(\mu) = 0$ has the form given in the first part of the proposition.

Now, let $\mu\in\sigma(A)$, then $\Delta(\mu) = 0$. It follows from the last equality that the functions φ, ψ can be written as

$$\varphi(x,\mu) = m_1(\mu)w(\mu,x), \quad \psi(x,\mu) = m_2(\mu)w(\mu,x) \qquad (A7)$$

where $m_1(\mu) \not\equiv 0$, $m_2(\mu) \not\equiv 0$, $w(\mu,x)$ are eigenfunctions of operator A corresponding to eigenvalue μ. The use of equation (A7) yields

$$\begin{aligned}\theta(\mu) &= -\mu^{-1}\int_0^l\int_0^l(\lambda r_x)_x g(x,\xi,\mu)\,\mathrm{d}\xi\,\mathrm{d}x\\ &= \mu^{-1}\left(\int_0^l\left(\psi(x,\mu)\int_0^x\varphi(\xi,\mu)b(\xi)\,\mathrm{d}\xi\right.\right.\\ &\quad\left.\left.+\int_x^l\varphi(x,\mu)\psi(\xi,\mu)b(\xi)\,\mathrm{d}\xi\right)(\lambda r_x)_x\,\mathrm{d}x\right)\\ &= -\mu^{-1}\int_0^l m_1m_2w(\mu,x)\int_0^l w(\mu,\xi)b(\xi)(\lambda r_x)_x\,\mathrm{d}x\\ &= -\mu^{-1}m_1m_2\langle w, Ar\rangle\langle c^{-1}b, w\rangle.\end{aligned}$$

Thus, when $\theta(\mu) = 0$, then one of the equalities should be fulfilled

$$\langle w, Ar\rangle = 0, \quad \langle c^{-1}b, w\rangle = 0. \qquad (A8)$$

On the other hand, it follows easily from the structure of equation (A1) and from the Fredholm alternative that, in the case when $\mu\in\sigma(A)$, the number μ enters into the spectrum F if and only if at least one of equalities (A8) holds.

(2) When $\mu\notin\sigma(A)$, relation (25) follows from equations (A2) and (A6) and when $\mu\in\sigma(A)$, it can be verified directly. An analogous method is used to prove relation (26).

METHODE D'INVERSION DES SYSTEMES DYNAMIQUES ET SON APPLICATION A LA RECUPERATION DES SOURCES INTERNES DE CHALEUR

Résumé—On développe la méthode d'inversion des systèmes dynamiques linéaires distribués qui fournit une solution analytique d'une certaine classe de problèmes inverses de conduction thermique. Comme illustration de l'approche générale, on considère le problème inverse de récupération de la composante variable au cours du temps d'une source interne de chaleur.

DAS VERFAHREN DER INVERSION DYNAMISCHER SYSTEME UND SEINE ANWENDUNG AUF DIE ERMITTLUNG INNERER WÄRMEQUELLEN

Zusammenfassung—Das Verfahren der Inversion linear verteilter dynamischer Systeme wird entwickelt. Hiermit ist es möglich, eine bestimmte Klasse inverser Wärmeleitprobleme qualitativ zu untersuchen und analytisch zu lösen. Beispielhaft wird das allgemeine Vorgehen anhand der Ermittlung der zeitlich veränderlichen Komponente einer inneren Wärmequelle gezeigt.

МЕТОД ОБРАТНЫХ ДИНАМИЧЕСКИХ СИСТЕМ И ЕГО ПРИМЕНЕНИЕ ДЛЯ ВОССТАНОВЛЕНИЯ ВНУТРЕННИХ ИСТОЧНИКОВ ТЕПЛА

Аннотация—Развивается метод обращения линейных распределенных динамических систем, позволяющий проводить качественное исследование и получить аналитическое решение определённого класса инверсных задач теплопроводности. В качестве иллюстрации к общему подходу рассмотрена обратная задача по восстановлению изменяющейся во времени компоненты внутреннего источника тепла.